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Point Symmetry Group of the Lagrangian

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The theory of finite point symmetry transformations is revisited within the frame of the general theory of transformations of Lagrangian mechanics. The point symmetry group $G(L)$ of a given Lagrangian function L (i.e., the Noether group) is thus obtained, and its main features are briefly discussed. The explicit calculation of the Noether group is presented for two rather simple c-equivalent Lagrangian systems. The formalism affords an introduction to the Noether theory of infinitesimal point symmetry transformations in Lagrangian mechanics; however, it is also of interest in its own right.

1. INTRODUCTION

In a previous paper (Aguirre and Krause, 1990) (hereafter referred to as paper I), a brief review of the theory of finite point transformations in Lagrangian mechanics is presented, which serves also as a general introduction for developing the theory of symmetry in mechanics. In the present paper we confine ourselves to some basic ideas leading to a unified theory of *finite point symmetries* in Lagrangian mechanics, within the conceptual framework adopted in paper I.

Although finite symmetry transformations are not usually associated with conservation laws [as are the infinitesimal symmetries of a system (Noether, 1918; see also Lutzky, 1979*a*,*b*, 1981; Hojman and Harleston, 1981)], they are of great interest in mechanics. In this sense, it is enough to recall, for instance, the important role played by finite rotations and displacements, or finite Galilei and Lorentz transformations, as symmetry transformations in physics. Hence, we deem the present study of *finite* symmetry transformations in Lagrangian mechanics to be an interesting subject in its own right.

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As we have seen in paper I, in order to have an adequate transformation theory for the Lagrange formalism, one has to enlarge the usual theoretical framework from the manifold of configuration space to that of configuration spacetime [(Trümper, 1983)], conceived as the background arena on which the relevant transformations locally act. In this fashion, the concept of the Lagrange group was introduced in that paper. As we shall see in the present paper, this enlarged background manifold is also necessary for a satisfactory group-theoretic analysis of point symmetries in Lagrangian mechanics. In fact, the concept of the *full point symmetry group of the Lagrangian* will be briefly discussed here, as an important subgroup of the general Lagrange group. This symmetry group is formed by all those *Lagrangian transformations* (cf. below) that have the property of maintaining invariant the functional form of the Lagrangian, and hence, it belongs to the group of dynamical symmetries of the system (Mariwalla, 1975).

It turns out that for most Lagrangian systems of mechanical interest, the point symmetry group appears as a finite-dimensional Lie group. The mechanical relevance of this group is mainly twofold. First, the associated Lie algebra is precisely the *Noether algebra* obeyed by the infinitesimal point symmetry generators of the Lagrangian function. Hence, this group can be also referred to as the *Noether group.* (This matter shall be studied in a forthcoming paper.) Second, the existence of the Noether group of a system settles the basis of the "special relativity" theory which is peculiar to that system. In fact, the Noether group contains *all* the coordinate transformations in configuration spacetime under which the Lagrangian behaves as an invariant function. In this sense one can speak, for instance, of the "special relativity" theory of the simple harmonic oscillator, or of the Kepler system, as one usually speaks of the special relativity (Galilean or Einsteinian) theory of a free-particle system (Mariwalla, 1975). This generalization is not purely rhetorical. The dynamical generalization implied in this concept is important because most Lagrangians can be characterized by a "special relativity" theory, in a rather peculiar fashion.

The organization of this paper is as follows: In Section 2 we present the point symmetry group of the Lagrangian function. Section 3 is devoted to a brief discussion of the problem set by the existence of c-equivalent Lagrangians. Finally, in Sections 4 and 5 we present two miscellaneous examples, namely, the Noether groups of the one-dimensional free particle, and that of the simple harmonic oscillator, respectively, which are explicitly calculated in these sections.

2. THE POINT SYMMETRY GROUP OF THE LAGRANGIAN

"Lagrangian transformations" have been discussed in paper I, as one of the basic tools of the transformation theory of mechanics. These transformations are defined as follows. One first considers a sufficiently smooth transformation of the variables $(t; q^1, \ldots, q^n)$ into a new set of variables $(T; Q^1, \ldots, Q^n)$ in configuration spacetime:

$$\begin{aligned} T &= T(t, q) \\ Q^j &= Q^j(t, q) \end{aligned} \tag{2.1}$$

which are given in some specified open connected region $R \subset \{(t, q)\}$, and are globally invertible on R. Then, if equations (2.1) are interpreted as a local *transformation of coordinates* in configuration spacetime, a new class of Lagrangian functions $\hat{L}$ can be defined by

$$\dot{T}\hat{L}(T, Q, \mathring{Q}) = KL(t, q, \dot{q}) + \dot{G}(t, q) \tag{2.2}$$

where L is the old Lagrangian, G an arbitrary gauge function, K an arbitrary constant, and $\mathring{Q}^j = dQ^j/dT$, with T denoting the new independent variable (cf. paper I for details). We call equation (2.2) a *Lagrangian transformation* of **L** induced by a local coordinate transformation in configuration spacetime.

Henceforth we use the same shorthand notation introduced in paper I. We write D to denote a general diffeomorphism in configuration spacetime, as defined in equation (2.1), and D^{-1} to denote its inverse. The symbolic expression $D_{21} = D_2 D_1$ denotes the composite diffeomorphism:

$$T = T_2[T_1(t, q), Q_1(t, q)]$$

and $Q^j = Q_2^j[T_1(t, q), Q_1(t, q)]$. In particular, we write I for the identity transformation: $T = t$ and $Q^j = q^j$. Accordingly, let us write the symbolic equation

$$\hat{L} = (D, K, G)L \tag{2.3}$$

to briefly denote a general Lagrangian transformation of L under D, according to the law stated in equation (2.2). In this manner, we can also write $\tilde{L} = (I, K, G)L$ as a symbol for a gauge transformation of L generated by K and G.

In paper I it was shown that the set of all Lagrangian transformations constitutes a group (under a particular law of combination). We call this group the *Lagrange group* for n-dimensional Lagrangian systems, and we denote it by $L_{(n)} = \{(D, K, G)\}$. This is the most general group of point transformations acting on the set $\{L\}$ (of all representative Lagrangians for n-dimensional systems) that keeps invariant the Lagrangian formalism of mechanics.

We now come to one of the main concepts of this subject. As a very special case of Lagrangian transformation, a coordinate transformation

$$\hat{t} = f(t, q)$$
$$\hat{q}^j = g^j(t, q) \tag{2.4}$$

is called a *point symmetry transformation* of the Lagrangian function $L(t, q, \dot{q})$ if, and only if, there exist a gauge function $\sigma(t, q)$ and a constant κ, such that

$$\dot{\hat{t}} L(\hat{t}, \hat{q}, \mathring{\hat{q}}) = \kappa L(t, q, \dot{q}) + \dot{\sigma}(t, q) \tag{2.5}$$

where $\mathring{\hat{q}}^j = d\hat{q}^j/d\hat{t}$. Note that the *symmetry gauge function* $\sigma(t, q)$ and the *symmetry scaling constant* κ are no longer arbitrary (i.e., they are unique), for they are fixed by the functional form of L and by the considered point symmetry transformation (2.4). Indeed, this definition says that a point symmetry of L is a coordinate transformation such that, when combined with a suitable gauge transformation, it keeps invariant the functional *form* of L in the sense of the general law (2.2).

Hence, using the same symbolic notation, let us also write the equation

$$L = (S, \kappa, \sigma)L \tag{2.6}$$

to briefly denote the Lagrangian symmetry transformation (2.5) of L, where S is the symbol for the associated symmetry coordinate transformation (2.4), and (I, κ, σ) denotes the unique gauge transformation that participates in (2.5). See also paper I, Section 5. Note that the identity $L = (I, 1, 0)L$ is a trivial instance of equation (2.6).] Henceforth, (S, κ, σ) will be referred to as a *point symmetry* of L.

The set of all point symmetries (S, κ, σ) of a given Lagrangian function is endowed with the group property under the same product law of the Lagrange group $L_{(n)}$ [cf. paper I, equation (5.7)], namely

$$(S_{21}, \kappa_{21}, \sigma_{21}) = (S_2, \kappa_2, \sigma_2)(S_1, \kappa_1, \sigma_1) = (S_2 S_1, \kappa_2\kappa_1, K_2\sigma_1 + \sigma_2) \tag{2.7}$$

and the inversion law reads $(S, \kappa, \sigma)^{-1} = (S^{-1}, \kappa^{-1}, -\kappa^{-1}\hat{\sigma})$, where $\hat{\sigma}(\hat{t}, \hat{q}) = \sigma(t, q)$. This group is called the *point symmetry group of the Lagrangian*, and we shall denote it by $G(L) = \{(S, \kappa, \sigma)\}$. Usually, $G(L)$ is a *finite-dimensional Lie group*, which acts locally in configuration spacetime through the symmetry diffeomorphisms S that characterize L (see also Section 4). In such case, we shall call $G(L)$ the *Noether group* of the system.

Each point symmetry group $G(L)$ is a subgroup of the Lagrange group $L_{(n)}$. However, a word of caution is necessary concerning the group structure of $G(L)$. Since every gauge transformation changes the form of the Lagrangian, one has $(I, \kappa, \sigma) \notin G(L)$, unless $\kappa = 1$ and $\sigma = 0$. Thus, the group $G_{(n)}$ of all gauge transformations is not a subgroup of $G(L)$, even if we restrict $G_{(n)}$ to those gauge transformations (I, κ, σ) which participate in the elements of $G(L)$. Technically, this means that $G(L)$ is *not* the direct product (neither is it the semidirect product) of the group of point symmetries of L [i.e., $S_{(n+1)}(L) = \{S\} \sim \{(S, 1, 0)\}$], and of the restricted gauge subgroup $\{(I, \kappa, \sigma)\}$ of $G_{(n)}$. In fact, one should underline that the following isomorphism holds:

$$G(L) \sim S_{(n+1)}(L) \tag{2.8}$$

because for each $S \in S_{(n+1)}(L)$ the corresponding gauge factor (I, κ, σ) that yields (S, κ, σ) is *unique* and, furthermore, the one-to-one relation $(S, \kappa, \sigma) \leftrightarrow S$ preserves the group structure. Hence, each group $G(L)$ affords a faithful realization of the corresponding group $S_{(n+1)}(L)$ which acts in configuration spacetime. In other words, the fixed gauge transformation (I, κ, σ) that appears in (S, κ, σ) is indispensable for having a symmetry of L, but it does not contribute to the group structure of $G(L)$, which is essentially the same as that of the group of diffeomorphisms $S_{(n+1)}(L)$.

It is a simple exercise to show that

$$\tilde{L} = (I, K, L)L \;\Rightarrow\; G(\tilde{L}) \sim G(L) \tag{2.9}$$

which means that all g-equivalent Lagrangians have essentially the same point symmetries.

3. THE PROBLEM OF c-EQUIVALENT LAGRANGIANS

As we have seen in paper I, the concept of g-equivalent (i.e., gauge-equivalent) Lagrangians plays a very important role in the transformation theory of mechanics. In this section we discuss another concept of "equivalent Lagrangians" which is also useful because it gives rise to a fundamental problem in Lagrangian mechanics. In that paper, the following definition has been introduced: two given Lagrangians L and $\hat{L}$ (for n-dimensional systems) are said to be *c-equivalent* (i.e., curve-equivalent) when there exist a particular coordinate transformation and a suitable gauge transformation which transform one Lagrangian function into the other, in the sense of equation (2.2).

According to this definition, the interesting problem arises: under what conditions does c-equivalence hold for two *given* (n-dimensional) Lagrangian functions? For instance, one would like to know under what circumstances one can transform a given system into a system of (say) uncoupled harmonic oscillators, by means of a special coordinate transformation combined with a suitable gauge transformation. This is a practical question indeed, because if one knows the allowed motions of the transformed system, the inverse transformation of coordinates would automatically solve the problem of motion of the given system. As we shall see, the answer to this fundamental question is intimately related with the point symmetry properties exhibited by the two given Lagrangians.

We next prove the following theorem: If S is a point symmetry of $L(t, q, \dot{q})$, then the conjugated diffeomorphism

$$\hat{S} = DSD^{-1} \tag{3.1}$$

is a point symmetry transformation for all those c-equivalent Lagrangians $\hat{L}(T, Q, \mathring{Q})$ obtained from $L(t, q, \dot{q})$ by the action of D.

The proof is as follows. By hypothesis, we have

$$L = (S, \kappa, \sigma)L \tag{3.2}$$

Let us then introduce a new Lagrangian function $\hat{L}$, given by the Lagrangian transformation

$$\hat{L} = (D, K, G)L \tag{3.3}$$

where D is the diffeomorphism $T = T(t, q)$ and $Q^j = Q^j(t, q)$. We now *define* the function

$$\hat{L}' = (D, K, G)(S, \kappa, \sigma)L \tag{3.4}$$

where D is the *same diffeomorphism* used in equation (3.3) [i.e., in equation (3.4), D stands for $\hat{T} = T(\hat{t}, \hat{q})$ and $\hat{Q}^j(\hat{t}, \hat{q})$, while $G(\hat{t}, \hat{q})$ is the *same function* $G(t, q)$ used in equation (3.3), although expressed in terms of new coordinates $(\hat{t}, q)$]. In this fashion, we see that

$$\hat{L}'(\hat{T}, \hat{Q}, \mathring{\hat{Q}}) = \hat{L}(\hat{T}, \hat{Q}, \mathring{\hat{Q}}) \tag{3.5}$$

since it is only the "naming" of the coordinates that differs between equations (3.3) and (3.4). Thus, we can write

$$\hat{L} = (D, K, G)(S, \kappa, \sigma)L \tag{3.6}$$

instead of equation (3.4), and so we get

$$\hat{L} = (D, K, G)(S, \kappa, \sigma)(D, K, G)^{-1}\hat{L} \tag{3.7}$$

from which the point symemtry of $\hat{L}$ follows immediately; i.e., we obtain

$$\hat{L} = (\hat{S}, \hat{\kappa}, \hat{\sigma})\hat{L} \tag{3.8}$$

where the point symmetry $(\hat{S}, \hat{\kappa}, \hat{\sigma})$ of $\hat{L}$ is given by the conjugacy

$$(\hat{S}, \hat{\kappa}, \hat{\sigma}) = (D, K, G)(S, \kappa, \sigma)(D, K, G)^{-1} \tag{3.9}$$

In this manner, we have $\hat{S}=DSD^{-1}$ [i.e., equation (3.1)], $\hat{\kappa}=\kappa$, and

$$\hat{\sigma}(T, Q)=K\sigma(t, q)-\kappa G(t, q)+G(\hat{t}, \hat{q}) \tag{3.10}$$

as the reader can check. This finishes the proof of the theorem.

Some consequences of this theorem follow. First, we observe that when two given Lagrangian functions are c-equivalent there is a one-to-one correspondence between their respective point symmetries [in fact, one has $\hat{S}=DSD^{-1} \leftrightarrow S=D^{-1}\hat{S}D$, where D is *unique*]. Moreover, the conjugacy (3.9) preserves the group structure of $G(L)$. So we have the following corollary: A necessary condition for c-equivalence of two given Lagrangian functions is that their point symmetry groups be isomorphic; briefly,

$$\hat{L}=(D, K, G)L \;\Rightarrow\; G(\hat{L})\sim G(L) \tag{3.11}$$

From a practical point of view, this means that it is not possible to transform a Lagrangian function into another (by means of a change of coordinates and a suitable gauge transformation) if they have *different* point symmetry properties. This answers (in part) the problem motivated by the notion of c-equivalent Lagrangians. However, it must be also mentioned here that the isomorphism $G(L)\sim G(\hat{L})$ is *not* a sufficient condition to ensure the c-equivalence of L and $\hat{L}$.

From a group-theoretic point of view, this means that the action of the Lagrange group $L_{(n)}$ on the set $\{L\}$ is *not transitive* (e.g., Michel, 1972, pp. 133–150). (In fact, there are many examples of Lagrangian functions, for n-dimensional systems, with nonisomorphic point symmetry groups and there is no element of $L_{(n)}$ available to "connect" such Lagrangians.) Hence, the set $\{L\}$ becomes decomposed into disjoint classes of c-equivalent Lagrangians, and the action of $L_{(n)}$ is transitive only within each c-class of Lagrangians. Furthermore, each c-class is characterized by a well-defined group, which becomes locally realized as the group $G(L)\sim S_{(n+1)}(L)$ for any function L that belongs to the given class. The general interest of these results for the Lagrangian theory of *interactions* is immediate.

4. POINT SYMMETRIES OF THE ONE-DIMENSIONAL FREE-PARTICLE LAGRANGIAN

We devote this section to the study of the point symmetry group $G(L_o)$ of a one-dimensional free particle:

$$L_o(\dot{q})=\tfrac{1}{2}\dot{q}^2 \tag{4.1}$$

[Besides its intrinsic interest, let us here recall that *all* one-dimensional Newtonian linear systems are c-equivalent to the system $\ddot{q}=0$ (Arnold, 1988, p. 44).]

Hence, we search for a coordinate transformation $\hat{t}=f(t, q)$ and $\hat{q}=g(t, q)$ such that

$$\dot{\hat{t}}\,\mathring{\hat{q}}^2=\kappa\dot{q}^2+2\dot{\sigma}(t, q) \tag{4.2}$$

holds for some (yet unknown) κ and $\sigma(t, q)$. Equation (4.2) can be written in the form of a third-order polynomial in $\dot{q}$, which must vanish identically for all values of $\dot{q}$. Thus, after some manipulations, we obtain the following system of nonlinear first-order partial differential equations:

$$\begin{aligned} f_q&=0\\ g_q^2-\kappa f_t&=0\\ g_t g_q-\sigma_q f_t&=0\\ g_t^2-2\sigma_t f_t&=0 \end{aligned} \tag{4.3}$$

The first three equations can be integrated directly, to read

$$\begin{aligned} &f=\rho(t)\\ &g=[\kappa\dot{\rho}(t)]^{1/2}q+\varphi(t)\\ &\sigma=[\kappa\ddot{\rho}(t)/4\dot{\rho}(t)]q^2+[\kappa/\dot{\rho}(t)]^{1/2}\dot{\varphi}(t)q+\mu(t) \end{aligned} \tag{4.4}$$

where ρ, φ, and μ are functions of t which one has to determine by substituting from equations (4.4) into the fourth equation in (4.3). This substitution yields a polynomial of the second order in q, which vanishes identically and, hence, the coefficients give the following equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\ddot{\rho}}{\dot{\rho}}\right)&=\frac{1}{2}\left(\frac{\ddot{\rho}}{\dot{\rho}}\right)^2\\ \frac{d}{dt}\left(\frac{\varphi}{\dot{\rho}^{1/2}}\right)&=\frac{\ddot{\rho}\dot{\varphi}}{2\dot{\rho}^{3/2}}\\ \dot{\mu}&=\frac{\dot{\varphi}^2}{2\dot{\rho}} \end{aligned} \tag{4.5}$$

which one easily integrates.

In this fashion, the most general solutions to equation (4.3) can be written in the form

$$\begin{aligned} \hat{t}&=\frac{k_1 t+k_2}{1-k_3 t}\\ \hat{q}&=\frac{k_4 q+k_5 t+k_6}{1-k_3 t} \end{aligned} \tag{4.6}$$

where the k's are six constants of integration and the associated symmetry gauge transformation (I, κ, σ) is given by

$$\begin{aligned} \sigma(t, q) &= \frac{[k_3k_4q + (k_5 + k_3k_6)]^2}{2k_3(k_1 + k_2k_3)(1 - k_3t)} + \sigma_0 \\ \kappa &= \frac{k_4^2}{k_1 + k_2k_3} \end{aligned} \tag{4.7}$$

where σ_0 is an arbitrary (unnecessary) constant of integration, provided that σ be is defined at $k_3 = 0$.

The arbitrary k's are *essential parameters* of a faithful (local) realization of a *six-dimensional Lie group* G_o; i.e., $G(L_o) \sim G_o$. The *identity element* of G_o corresponds to the choice $k_1 = 1$, $k_2 = k_3 = 0$, $k_4 = 1$, $k_5 = k_6 = 0$, and the *group multiplication law* in G_o is given by

$$\begin{aligned} k_1'' &= \frac{k_1'k_1 - k_2'k_3}{1 - k_3'k_2} \\ k_2'' &= \frac{k_2' + k_1'k_2}{1 - k_3'k_2} \\ k_3'' &= \frac{k_3'k_1 + k_3}{1 - k_3'k_2} \\ k_4'' &= \frac{k_4'k_4}{1 - k_3'k_2} \\ k_5'' &= \frac{k_5'k_1 - k_6'k_3 + k_2'k_5}{1 - k_3'k_2} \\ k_6'' &= \frac{k_6' + k_5'k_2 + k_4'k_6}{1 - k_3'k_2} \end{aligned} \tag{4.8}$$

from which one obtains the *group inversion law*, as well as the *associative law*, for the parameters of $G(L_o)$.

One easily obtains the one-parameter subgroups of $G(L_o)$. The *homogeneous Galilei group* (in two-dimensional spacetime) is a subgroup of $G(L_o)$, characterized by the "boost parameter" k_5. We observe that the group of *affine transformations* of the real line $\{-\infty < q < \infty\}$ is also a subgroup of $G(L_o)$, which is realized by the following diffeomorphism in two-dimensional spacetime: $\hat{t} = t$, $\hat{q} = k_4q + k_6$, with $\sigma = 0$, and $\kappa = k_4^2$.

Another interesting (five-parameter) subgroup of $G(L_o)$ can be found if one sets $\kappa = 1$ in equation (4.2) to begin with. The reader can convince herself or himself that with this choice for κ one gets $k_4=(k_1+k_2k_3)^{1/2}$ in the general solution given by equations (4.6) and (4.7), as consistency demands.

Let us here clarify some subtle points concerning the constant κ. First, note that even when one has $\kappa\neq 1$, one does *not* get

$$L(t, q, \dot{q}) \to \kappa L(\hat{t}, \hat{q}, \mathring{\hat{q}}),$$

as one would think on first sight. Rather, one gets $L(t, q, \dot{q}) \to L(\hat{t}, \hat{q}, \mathring{\hat{q}})$; that is, one has a symmetry of the Lagrangian, *sensu stricto*, even when it turns out that $\kappa\neq 1$. Second, we note that $\kappa\neq 1$ does not appear only as a consequence of a trivial *change of scale* in t or in q (or in both); more involved symmetry transformations of spacetime coordinates also can be related with this result. Finally, let us observe that if *one imposes* the condition $\kappa=1$ from the beginning, one usually obtains a *subgroup* of the full point symmetry group $G(L)$ of the given Lagrangian function L.

5. THE NOETHER GROUP OF A SIMPLE HARMONIC OSCILLATOR

As we have already observed, the foregoing example has far-reaching consequences because of the c-equivalence property of all one-dimensional linear systems. For instance, it is well known that the local coordinate transformation

$$\begin{aligned} T &= \tan \omega t \\ Q &= q \sec \omega t \end{aligned} \tag{5.1}$$

reduces the equation of motion $\ddot{q}+\omega^2 q=0$ to the free particle equation $Q=0$. In fact, if we consider the Lagrangian $L_o(\mathring{Q})=\frac{1}{2}\mathring{Q}^2$ under this diffeomorphism, after some manipulations we obtain

$$\tfrac{1}{2}\dot{T}\mathring{Q}^2=\frac{1}{2\omega}(\dot{q}^2-\omega^2 q^2)+\frac{d}{dt}(\tfrac{1}{2}q^2 \tan \omega t) \tag{5.2}$$

(cf. paper I), where we identify $K=\omega^{-1}$ and $G(t, q)=\frac{1}{2}q^2 \tan \omega t$. This means that the Lagrangians $L_o=\frac{1}{2}\dot{q}^2$ and $L=\frac{1}{2}(\dot{q}^2-\omega^2 q^2)$ belong to the same c-class, and therefore one has $G(L)\sim G(L_o)\sim G_o$.

Hence, in order to obtain the group $G(L)$, we can proceed as follows. Recalling equations (4.6), and using equation (5.1), let us write the diffeomorphism

$$\hat{T}=\frac{k_1T+k_2}{1-k_3T}=\tan\omega\hat{t}=\frac{k_1\tan\omega t+k_2}{1-k_3\tan\omega t} \tag{5.3a}$$

$$\hat{Q}=\frac{k_4Q+k_5T+k_6}{1-k_3T}=\hat{q}\sec\omega\hat{t}=\frac{k_4q\sec\omega t+k_5\tan\omega t+k_6}{1-k_3\tan\omega t} \tag{5.3b}$$

the meaning of which is clear. Then, after some simple steps, we obtain (Aguirre and Krause, 1987, 1988*a*,*b*)

$$\sin\omega\hat{t}=\frac{k_1\sin\omega t+k_2\cos\omega t}{[(k_1\sin\omega t+k_2\cos\omega t)^2+(\cos\omega t-k_3\sin\omega t)^2]^{1/2}} \tag{5.4a}$$

$$\hat{q}=\frac{k_4q+k_5\sin\omega t+k_6\cos\omega t}{[(k_1\sin\omega t+k_2\cos\omega t)^2+(\cos\omega t-k_3\sin\omega t)^2]^{1/2}} \tag{5.4b}$$

In the same manner, from equations (3.10) and (4.7) we get the associated gauge quantities for the realization of $G(L)$. (We leave this task to the reader.) Equations (5.4) entail the point symmetries of the standard Lagrangian $L=\frac{1}{2}(\dot{q}^2-\omega^2q^2)$. In fact, they correspond to a new *faithful* realization of the same Lie group G_o whose abstract structure in terms of the group parameters is exhibited in equations (4.8). Thus, we have obtained the Noether group of the simple harmonic oscillator, which is the largest group of spacetime coordinate transformations endowed with the property of keeping invariant the standard Lagrangian of the system. The infinitesimal symmetry transformations obtained from equations (5.4) yield the Noether theory of the simple harmonic oscillator.

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REFERENCES

Aguirre, M., and Krause, J. (1987). *Journal of Physics A: Mathematical and General*, **20**, 3553.
Aguirre, M., and Krause, J. (1988*a*). *Journal of Mathematical Physics*, **29**, 9.
Aguirre, M., and Krause, J. (1988*b*) *Journal of Mathematical Physics*, **29**, 1746.
Aguirre, M., and Krause, J. (1990). *International Journal of Theoretical Physics*, submitted.
Arnold, V. I. (1988). *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York.
Hojman, S., and Harleston, H. (1981). *Journal of Mathematical Physics*, **22**, 1414.
Lutzky, M. (1979*a*). *Physics Letters*, **72A**, 86.
Lutzky, M. (1979*b*). *Physics Letters*, **75A**, 8.

Lutzky, M. (1981). *Journal of Mathematical Physics*, **22**, 1626.
Mariwalla, K. (1975). *Physics Reports*, **20**, 287.
Michel, L. (1972). In *Statistical Mechanics and Field Theory*, R. N. Sen and C. Weil, eds., Israel University Press, Jerusalem.
Noether, E. (1918). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, **2**, 235.
Trümper, M. (1983). *Annals of Physics*, **149**, 203.